# CORRELATORS AND DESCENDANTS OF SUBCRITICAL STEIN MANIFOLDS

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Abstract. We continue the study of the contact homology of subcritical Stein manifolds initiated by Mei-Lin Yau. With the technical assumption that the first Chern class of the Stein domain vanishes, we determine the full contact homology of the boundary of a subcritical Stein domain. Moreover we calculate the genus 0 correlators and descendants of one marked point for the Stein domain. As an application, we prove that if a Kähler manifold $M^{2n}$ admits a subcritical polarization and $c_1(M)$ is proportional to the Kähler form, then $M$ is uniruled.

## 1. Introduction

An open complex manifold $(M^{2n}, J)$ is *Stein* if it can be realized as a properly embedded complex submanifold of some $\mathbb{C}^N$. In this paper we will assume $n \geq 3$. A smooth function $f : M \to \mathbb{R}$ is *exhausting* if it is proper and bounded from below. Let $d^J f$ denote $df \circ J$. The function $f$ is *plurisubharmonic* if the associated 2-form $\omega_f = -dd^J f$ is a symplectic form taming $J$, i.e., $\omega_f(v, Jv) > 0$ for every non-zero tangent vector $v$. Plurisubharmonicity is an open condition. Therefore $f$ can be assumed to be Morse. By a theorem of Grauert, an open complex manifold is Stein if and only if it admits a plurisubharmonic function.

A Stein manifold $(M^{2n}, J)$ with an exhausting plurisubharmonic function $f$ admits the following associated structures:

- a symplectic form $\omega_f = -dd^J f$, $\omega_f$ is $J$-invariant,
- a primitive $\alpha = -d^J f$,
- a vector field $Y$ such that $\alpha = \iota_Y \omega$,
- a metric $g(v, w) = \omega(v, Jw)$.

Since $L_Y \omega = \iota_Y d\omega + d(\iota_Y \omega) = d\alpha = \omega$, the vector field $Y$ is Liouville, i.e., the flow of $Y$ acts by expanding the symplectic form. In fact $Y$ is the gradient vector field of $f$ with respect to the metric $g$,

$$df = -df \circ J \circ J = \alpha \circ J = (\iota_Y \omega) \circ J = \iota_Y g.$$

By rescaling $f$ we can assume that $Y$ is a complete vector field. If the Morse function $f$ has finitely many critical points, then $M$ is of *finite type*. If the Morse indices of the critical points are all less than $n$, then $M$ is *subcritical*.

A Stein manifold $(M, \omega_f)$ of finite type can be viewed as a symplectic cobordism with one positive end in the sense of [EGH00]. Suppose $V$ is a hypersurface transverse to $Y$ which encloses all the critical points of $f$, then $V$ is of contact type with contact form $\alpha|_V$. Using the flow of $Y$, we see that the open end of $M$ is symplectomorphic to $(V \times [0, \infty), d(e^t \alpha|_V))$.

The framework of symplectic field theory provides symplectic invariants of $M$. The goal of this paper is to compute some of these invariants. The definitions of the invariants will be given in the next section.

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In [Yau04], Mei-Lin Yau showed that if $M$ is subcritical and $c_1(M) = 0$, then the *cylindrical contact homology* of $V$, $HC^{\mathsf{cyl}}(V)$, is isomorphic as a graded vector space to the direct sum $\bigoplus_{i=1}^{\infty} H_*(M, \partial M)[2i-4]$, where $H_*(M, \partial M)[2i-4]$ is a copy of the relative homology with a positive degree shift of $2i-4$.

Note that for a subcritical Stein manifold, $c_1(M) = 0$ is equivalent to $c_1(\xi) = 0$, where $\xi$ is the contact distribution on $V$. The unstable submanifolds of $M$ are of codimension at least 4, and the complement of the unstable manifolds is homeomorphic to $V \times \mathbb{R}$, so $c_1(M) = c_1(TM|_V)$. But $TM|_V = \xi \oplus \mathbb{C}$ where $\mathbb{C}$ is spanned by the Reeb field and the Liouville field, thus $c_1(M) = c_1(\xi)$.

Our first result determines the *full contact homology algebra* of $M$, $HC(V)$.

For a graded vector space $W = \oplus W_i$, let $\Lambda(W)$, the graded exterior algebra of $W$, be the quotient of the tensor algebra $W$ by the ideal generated by the graded commuting relations.

**Theorem 1.1.** *If $M$ is a subcritical Stein manifold and $c_1(M) = 0$, then*

$$HC(V) \cong \Lambda\left(HC^{\mathsf{cyl}}(V)\right).$$

In [Yau04], if a suitable plurisubharmonic function $f$ and a contact form $\alpha$ are chosen, then $\mathcal{C}^\alpha$, the chain complex for $HC^{\mathsf{cyl}}(V, \alpha)$, can be identified with copies of $\mathrm{Crit}(f)$, the Morse complex of a gradient–like vector field on $M$, by a chain map $\Psi(\alpha, f)$. This chain map $\Psi(\alpha, f)$ induces the desired isomorphism on homology.

However the naturality of this isomorphism was unknown. More precisely, if $g$ is another function and $\beta$ another contact form, then there are natural chain maps

$$\begin{aligned} \Phi(\alpha,\beta) : \mathcal{C}^\alpha &\to \mathcal{C}^\beta \\ \Theta(f,g) : \mathrm{Crit}(f) &\to \mathrm{Crit}(g) \end{aligned}$$

which induce isomorphisms on $HC(V)$ and $H_*(M, \partial M)$ respectively. Does the following diagram commute at homology level?

$$\begin{tikzcd} \mathcal{C}^\alpha \arrow[r, "\Psi(\alpha,f)"] \arrow[d, "\Phi(\alpha,\beta)"] & \mathrm{Crit}(f) \arrow[d, "\Theta(f,g)"] \\ \mathcal{C}^\beta \arrow[r, "\Psi(\beta,g)"] & \mathrm{Crit}(g) \end{tikzcd}$$

By a computation of one-point descendant correlators of genus 0, we give an affirmative answer to the above question.

**Theorem 1.2.** *There exists a non-degenerate pairing*

$$GW : HC^{\mathsf{cyl}}(V) \otimes \bigoplus_{i=1}^{\infty} H^*(M, \partial M)[i] \longrightarrow \mathbb{R}.$$

*The map $GW$ induces an isomorphism $HC^{\mathsf{cyl}}(V) \to \bigoplus_{i=1}^{\infty} H_*(M, \partial M)[i]$ which coincides with $\Psi(\alpha, f)$ for any choice of $f$ and $\alpha$.*

In [BC01], Biran and Cieliebak studied *subcritical polarization* of a Kähler manifold, which is a Kähler manifold $(M, \omega, J)$, where $\omega$ is an integral Kähler form, together with a smooth reduced complex hypersurface $\Sigma$ representing the Poincaré dual of $k[\omega]$, such that the complement $M \setminus \Sigma$ is a subcritical Stein manifold. Biran and Cieliebak asked if manifolds admitting subcritical polarizations are always uniruled. Combining our results on correlators on the subcritical Stein manifold and a standard Morse–Bott computation of correlators on the normal bundle of $\Sigma$, we obtain a partial answer to this question:

**Theorem 1.3.** *If $M$ admits a subcritical polarization and $c_1(M)$ is proportional to the Kähler form $\omega$, then $M$ is uniruled.*

The cylindrical contact homology of $V$ can be computed as the boundary of a subcritical Stein domain, and also as the unit normal bundle inside the normal bundle of $\Sigma$. Equating the two we obtain a set of relations for the Betti numbers of $\Sigma$ and $M \setminus \Sigma$, described below.

**Theorem 1.4.** *Suppose $(M^{2n}, J, \Sigma)$ is a subcritical polarization such that $c_1(M)$ is proportional to $\omega$. Let $a_i$ be the $i$-th Betti number of $M \setminus \Sigma$, $b_i$ the $i$-th Betti number of $\Sigma$, and $D$ the homological dimension of $M \setminus \Sigma$, i.e., the grading of highest non-vanishing homology group. Then*

**(a):** *the sequence $\{a_i\}$ is symmetric about $\frac{D}{2}$, i.e., $a_i = a_{D-i}$;*
**(b):** *for $2i < n$ or $2i+1 < n$,*

$$b_{2i} = \sum_{j=0}^{i} a_{2j}, \quad b_{2i+1} = \sum_{j=0}^{i} a_{2j+1}.$$

The computation of correlators and descendants relies on the fact that there is an $S^1$-action on a subcritical Stein manifold. In the spirit of localization theorems of Atiyah–Bott [AB84]and Graber–Pandharipande [GP00], the correlators and descendants can be determined by studying the fixed loci of the $S^1$-action.

This paper is organized as follows: in section 2 we review some basic facts about Stein manifolds and symplectic field theory, and define the relevant invariants; section 3 describes subcritical Stein manifolds and Reeb dynamics in more detail, essentially summarizing the previous work of Yau; in section 4 we determine the set of $S^1$-invariant holomorphic curves; in section 5 we prove the necessary transversality results; finally in the last section we apply our results to subcritical polarizations.

## 2. Stein manifolds and symplectic field theory

From now on we will restrict ourselves to Stein manifolds of finite type. All plurisubharmonic functions are assumed to be Morse unless otherwise stated. By a theorem of Eliashberg and Gromov, a Stein manifold carries a canonical symplectic structure. A different choice of plurisubharmonic function corresponds to a different choice of complete Liouville vector field $Y$ on the same symplectic manifold.

**Theorem 2.1** ([EG91])**.** *Let $f$ and $g$ be two plurisubharmonic functions which induce complete Liouville vector fields on a Stein manifold $M$. Then the manifolds $(M, \omega_f)$ and $(M, \omega_g)$ are symplectomorphic.*

It is easy to check that the unstable submanifolds of $f$ are isotropic, hence the Morse index of each critical point is less than or equal to $n$.

A Stein manifold is *split* if it is of the form $(M' \times \mathbb{C}, J' \times i)$, where $(M', J')$ is Stein. For a split Stein manifold we will always use a plurisubharmonic function of the form $f = f' + \kappa(x_n^2 + y_n^2)$, where $f'$ is a plurisubharmonic function on $M'$, $(x_n, y_n)$ the Euclidean coordinate of $\mathbb{C}$, and $\kappa$ a positive constant. The associated structures of $M$ are related to the associated structures of $M'$ as follows:

- symplectic form $\omega = \omega' + dx_n dy_n$,
- Liouville field $Y = Y' + \frac{1}{2}(x_n \partial_{x_n} + y_n \partial_{y_n})$,
- primitive $\alpha = \alpha' + \frac{1}{2}(x_n dy_n - y_n dx_n)$.

Note that $Y$ and $Y'$ share the same critical points.

A hypersurface $V' \subset M'$ is transverse to $Y'$ and encloses the critical points of $Y'$ if it can be realized as the level set of a function $\phi$ such that $Y'$ is gradient–like for $\phi$, and $\phi(p) < c$ for every critical point $p$ of $Y'$. Let $V \subset M$ be the level set $\{\phi + \kappa(x_n^2 + y_n^2) = c\}$. Such a $V$ is transverse to $Y$ and encloses the critical points of $Y$. It is said to be a *stabilization* of $V'$.

By construction, all structures and the contact type hypersurface $V$ admit an $S^1$-symmetry: rotation in the $\mathbb{C}$ component.

Two Stein structures $(M, J_0)$ and $(M, J_1)$ are *Stein homotopic* if there is a continuous family of Stein structures $(M, J_t)$ with exhausting plurisubharmonic functions $f_t$ such that the critical points of $f_t$ stay in some compact subset during the homotopy. Two Stein manifolds $(M_0, J_0)$ and $(M_1, J_1)$ are *deformation equivalent* if there exists a diffeomorphism $\phi : M_0 \to M_1$ such that $(M_0, J_0)$ and $(M_0, \phi^* J_1)$ are Stein homotopic.

**Theorem 2.2** ([Cie02])**.** *Every subcritical Stein manifold is deformation equivalent to a split one.*

If $(M_0, J_0)$ and $(M_1, J_1)$ are deformation equivalent, then $(M_0, \omega_{f_0})$ and $(M_1, \omega_{f_1})$ are symplectomorphic. Hence we always treat a subcritical Stein manifold as split.

A *symplectic cobordism* is an open symplectic manifold $(M, \omega)$ with cylindrical ends, i.e., $(M, \omega)$ can be decomposed into three parts, $M = M^+ \cup M' \cup M^-$, with the following properties:

- $M^+$ is symplectomorphic to $(V^+ \times (-\epsilon, \infty), d(e^t \alpha^+))$, where $(V^+, \alpha^+)$ is a closed contact manifold,
- $M'$ is compact,
- $M^-$ is symplectomorphic to $(V^- \times (-\infty, \epsilon), d(e^t \alpha^-))$, where $(V^-, \alpha^-)$ is a closed contact manifold.

A Stein manifold with a plurisubharmonic function $f$ is therefore a symplectic cobordism with one positive end.

Symplectic field theory invariants of a cobordism arise from the structure of the moduli spaces of finite energy holomorphic curves. We will give an extremely quick overview and introduce the invariants we will compute. See [EGH00] for a more complete discussion.

An almost complex structure $J$ on $M$ is *compatible* with the cobordism if it satisfies the following:

- $\omega(v, Jv) > 0$ for non-trivial $v \in TM$,
- $J$ is invariant under translation in the $t$ direction on the cylindrical ends $M^\pm$,
- $J\partial_t$ is the Reeb field on $V^\pm$,
- $J$ is an anti-involution on the contact distributions $\xi^\pm$.

Unfortunately the integrable complex structure $J$ of the Stein manifold is not compatible with the end structure $V \times (-\epsilon, \infty)$.

If the closed Reeb orbits on $V^\pm$ are non-degenerate, i.e., 1 is not an eigenvalue of the linearized returned map of the Reeb flow. Hofer, Wysocki and Zehnder [HWZ96] showed that finite energy $J$-holomorphic surfaces are asymptotic to closed Reeb orbits as $t \to \pm\infty$.

The symplectization of a contact manifold $(V, \alpha)$, $(M, \omega) = (V \times \mathbb{R}, d(e^t \alpha))$, can be considered as a trivial cobordism. For the trivial cobordism, we require compatible complex structures to be $\mathbb{R}$-invariant. Holomorphic curves then come in one parameter families. Suppose the first Chern class of the contact distribution $\xi$ vanishes. Then there is a consistent way to assign a Conley-Zehnder index $\mu_\gamma$ to each contractible Reeb orbit $\gamma$, such that the expected dimension of the moduli space $\mathcal{M}_{g,\{\gamma_1^+ \ldots, \gamma_a^+\},\{\gamma_1^- \ldots, \gamma_b^-\},m}$ is

$$\sum_i \mu_{\gamma_i^+} - \sum_j \mu_{\gamma_j^-} + (n-3)(2 - 2g - a - b) + 2m$$

where $\mathcal{M}_{g,\{\gamma_1^+\ldots,\gamma_a^+\},\{\gamma_1^-\ldots,\gamma_b^-\},m}$ is the moduli space of $J$-holomorphic maps from a genus $g$ surface with $a+b$ punctures and $m$ interior marked points to $M$, such that the $a$ positive punctures are asymptotic to $\{\gamma_i^+\}$ and the $b$ negative punctures to $\{\gamma_j^-\}$.

For coherent orientations of the moduli spaces, we only consider *good* orbits. An orbit is good if it is not an even multiple cover of another orbit whose linearized return map has an odd number of eigenvalues in $(-1,0)$.

Invariants of $(V,\xi)$ can be obtained by counting holomorphic curves in the symplectization of $V$.

The *full contact homology algebra* of $V$, $HC(V)$, is defined to be the homology of the chain complex of the graded commutative algebra generated by good contractible Reeb orbits. The differential of an orbit is given by the count of genus zero holomorphic curves with one positive puncture and any number of negative punctures:

$$\partial\gamma = \sum_i c_i\sigma_i,$$

where each $\sigma_i$ is a monomial $\beta_1\cdots\beta_{k_i}$, and $c_i$ is the algebraic number of expected dimension 1 (to account for the $\mathbb{R}$-invariance) moduli spaces of genus zero curves with a positive puncture asymptotic to $\gamma$ and several negative punctures asymptotic to $\beta_1,\ldots,\beta_{k_i}$. The differential is then extended to the entire algebra by Leibniz rule. From the study of the possible boundaries of an expected dimension 2 moduli space of genus zero curves with one positive end and several negative ends, one can deduce $\partial^2=0$.

A differential counting only holomorphic cylinders can also be defined on the additive group generated by the good contractible Reeb orbits. Define

$$\partial\gamma = \sum_i c_i\beta_i,$$

where $c_i$ is the algebraic number of expected dimension 1 moduli spaces of holomorphic cylinders from $\gamma$ to $\beta_i$. It was pointed out in [EGH00] that if there are no index $-1$, 0 or 1 holomorphic planes, then $\partial^2=0$. The resulting homology is called the *cylindrical contact homology* of $(V,\xi)$.

These homologies are invariants of the contact structure $\xi$ on $V$, independent of choices of contact form or complex structures.

With respect to a fixed Liouville vector field $Y$, it is clear that any two contact–type hypersurfaces enclosing the critical points of $Y$ are contactomorphic, since a flow along $Y$ takes one to the other. However this is no longer obvious for two different Liouville vector fields coming from two different plurisubharmonic functions. Still, the full contact homology and the cylindrical contact homology (if well defined) of the two surfaces are naturally isomorphic.

**Theorem 2.3.** *Let $Y_1$ and $Y_2$ be two complete Liouville vector fields on a symplectic manifold $M$, and $V_1$, $V_2$ be contact type hypersurfaces at infinity with respect to each. Then $HC(V_1)\cong HC(V_2)$, and $HC^{\mathsf{cyl}}(V_1)\cong HC^{\mathsf{cyl}}(V_2)$ if defined.*

*Proof.* Fix $V_1$, since the negative side of $V_1$ is compact, we can translate $V_2$ along $Y_2$ to $V_2'$ such that $V_2'$ lies entirely on the positive side of $V_1$. The region between $V_1$ and $V_2'$ then becomes a symplectic cobordism with two cylindrical ends. This induces a homomorphism $\Phi: HC(V_2)\to HC(V_1)$. Now translate $V_1$ along $Y_1$ to $V_1''$ such that $V_1''$ lies entirely on the positive side of $V_2'$. Similarly this induces $\Psi: HC(V_1)\to HC(V_2)$. Since the total region between $V_1''$ and $V_1$ is just part of a symplectization, by the composition property of symplectic cobordisms, $\Phi\circ\Psi$ is identity. Similarly $HC^{\mathsf{cyl}}(V_1)\cong HC^{\mathsf{cyl}}(V_2)$. □

Thus the full contact homology and cylindrical contact homology (if defined), are invariants the Stein manifold $M$.

We can also count holomorphic curves in $M$, the whole cobordism. Let $\overline{M}$ denote the compactification of $M$ where a copy of $V$ is attached "at infinity". Let $\mathcal{M}_{0,\{\gamma_1,\ldots,\gamma_n\},m}$ be the moduli space of genus 0 holomorphic curves with $n$ positive punctures asymptotic to Reeb orbits $\{\gamma_i\}$, and $m$ interior marked points.

In general the moduli spaces are not of the expected dimension. A regularization procedure must be performed to obtain a perturbed moduli space $\mathcal{M}^{virt}$, which is a weighted branched manifold with boundary. Together with the evaluation maps at the marked points, this represents a chain in $\left(\overline{M}^m, \partial\left(\overline{M}^m\right)\right)$.

One may integrate compactly supported forms over such chains, in other words, evaluate the integral

$$\int_{\mathcal{M}^{virt}} ev_1^*(\theta_1)\ldots ev_m^*(\theta_m)$$

where $\{\theta_i\}$ is a set of compactly supported closed forms on $M$. This is call a *correlator*. However unlike the case of Gromov–Witten invariants for closed manifolds, these chains are not cycles, they can have codimension one strata in the interior of $\overline{M}^m$. Therefore the value of a correlator depends on actual forms instead of their cohomology classes. To have an invariant correlator we look for moduli spaces whose union is a relative cycle.

**Theorem 2.4.** *Suppose $\sigma = [\sum_i c_i\sigma_i]$ is an element in the full contact homology of $V$, where each $\sigma_i = \gamma_{i_1}\cdots\gamma_{i_k}$ is a monomial. For each monomial $\sigma_i$, let $\mathcal{M}_i$ be the moduli space of $k_i$ holomorphic planes asymptotic to $\gamma_{i_1},\ldots,\gamma_{i_k}$ respectively, with $m$ marked points distributed between them in total. Then the union of the moduli spaces $\mathcal{M}_i$ with the evaluation map is a relative cycle in $\left(\overline{M}^m, \partial\left(\overline{M}^m\right)\right)$.*

*Proof.* This follows from compactness results on holomorphic curves in cobordisms. The boundary of a moduli space $\mathcal{M}_i$ that evaluates to the interior of $\overline{M}^m$ consists of multi-story holomorphic curves. The codimension one stratum consists of 2-story holomorphic curves. The curve in the symplectization of $V$ level has expected dimension one, contains no marked points, and consists of trivial cylinders $\gamma_{i_j}\times\mathbb{R}$ except for one component $i_0$, where it is a genus zero curve with one positive puncture asymptotic to $\gamma_{i_0}$ and any number of negative punctures. This is precisely what the rational contact homology differential counts. The fact that $[\sum_i \sigma_i]$ is a cycle in full contact homology implies that the codimension one strata from all the $\mathcal{M}_i$'s cancel each other. Therefore their union is a relative cycle in $(\overline{M}^m, \partial(\overline{M}^m))$. □

We will denote this cycle by $\mathcal{M}_{[\sigma]}$. The relative homology class of $\mathcal{M}_{[\sigma]}$ is independent of the contact form $\alpha$, as well as all other choices.

Another class of forms we can integrate over the moduli space $\mathcal{M}^{virt}$ are the $\psi$-classes. Suppose $p$ is a marked point, then for each element of the moduli space, the cotangent line to the domain curve at the marked point $p$ is a complex dimension one vector space. These spaces patch together to form a line bundle $L_p$, called a tautological line bundle of the moduli space. Define $\psi_p = c_1(L_p)$. An mixed integral of pull–backed forms and powers of $\psi$-classes over $\mathcal{M}^{virt}$ is called a *descendant*.

## 3. Subcritical Stein manifolds and Reeb dynamics

This section is essentially a summary of [Yau04]. Given a plurisubharmonic Morse function $f$ on a Stein manifold, or more importantly, the complete gradient–like Liouville vector field $Y$, we can reconstruct $M$ by attaching standard handles as in [Wei91]. We will describe the handle attachment procedure and discuss how the shape of the handle can be chosen to control the Reeb dynamics.

An index $k$ handle of real dimension $2n$ is modeled on the complex $n$–dimensional space $\mathbb{C}^n$ with the standard symplectic form $\omega_{\mathrm{st}}$ together with a standard complete Liouville vector field $Y_{\mathrm{st}}$. Let $(x_i, y_i)$ be the Euclidean coordinates, $\omega_{\mathrm{st}} = \sum_{i=1}^n dx_i \wedge dy_i$.

Define

$$Y_{\mathrm{st}} := \sum_{i=1}^k \left(2x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}\right) + \sum_{j=k+1}^n \frac{1}{2}\left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right),$$

$Y_{\mathrm{st}}$ is the gradient vector field of the function

$$f_{\mathrm{st}} = \sum_{i=1}^k (x_i^2 - \frac{1}{2} y_i^2) + \sum_{j=k+1}^n \frac{1}{4}(x_j^2 + y_j^2)$$

with respect to the Euclidean metric, and it is easy to see that $L_{Y_{\mathrm{st}}}\omega_{\mathrm{st}} = \omega_{\mathrm{st}}$.

Let the 1-form $\alpha_{\mathrm{st}}$ be the contraction $\iota_{Y_{\mathrm{st}}}\omega_{\mathrm{st}}$, it restricts to a contact 1-form on any hypersurface $V$ transverse to $Y_{\mathrm{st}}$.

$$\alpha_{\mathrm{st}} = \sum_{i=1}^k (2x_i dy_i + y_i dx_i) + \sum_{j=k+1}^n \frac{1}{2}(x_j dy_j - y_j dx_j).$$

We have an isotropic $k$-disk

$$D_{\mathrm{st}} = \left\{ \sum_{i=1}^k y_i^2 \leq 1, x_i = x_j = y_j = 0 \right\},$$

and its boundary sphere

$$S_{\mathrm{st}} = \left\{ \sum_{i=1}^k y_i^2 = 1, x_i = x_j = y_j = 0 \right\}.$$

Note that $S_{\mathrm{st}}$ lies on the hypersurface $V_- = \{f_{\mathrm{st}} = -1/2\}$. In fact $S_{\mathrm{st}}$ is an isotropic sphere of the contact manifold $V_-$.

Suppose we have a symplectic manifold $(M, \omega)$ with convex boundary, i.e., there is a local Liouville vector field $Y$ such that $\partial M$ is transverse to $Y$. If $S$ is an isotropic $k-1$ sphere on $\partial M$ together with a trivialization of the symplectic subnormal bundle, then using the following standard neighborhood theorem, we can attach a tubular neighborhood $U$ of $D_{\mathrm{st}}$ to $M$, identifying $S_{\mathrm{st}}$ with $S$:

**Theorem 3.1** ([Wei91])**.** *For $i = 1, 2$ let $(M_i, \omega_i, Y_i, V_i, S_i)$ be symplectic manifolds $(M_i, \omega_i)$ with Liouville vector fields $Y_i$, hypersurfaces $V_i$ transverse to $Y_i$, and isotropic submanifolds $S_i$ of $V_i$. Given a diffeomorphism from $S_1$ to $S_2$ covered by an isomorphism between their symplectic subnormal bundles, there exist neighborhoods $N_i$ of $S_i$ in $M_i$ and a symplectomorphism between them extending the given mapping on $Y_1$, such that $(N_1, \omega_1, Y_1, V_1, S_1)$ is taken to $(N_2, \omega_2, Y_2, V_2, S_2)$.*

Apply this theorem to the pair $(M, \omega, Y, \partial M, S)$ and $(\mathbb{C}^n, \omega_{\mathrm{st}}, Y_{\mathrm{st}}, V_-, S_{\mathrm{st}})$. Note that the symplectic normal bundle of $S_{\mathrm{st}}$ is trivial and has a natural framing

$$\left\{\partial_{x_{k+1}}, \partial_{y_{k+1}}, \ldots, \partial_{x_n}, \partial_{y_n}\right\}.$$

The Liouville vector field $\widetilde{Y}$ on the new manifold $M \cup U$ restricts to the $Y$ on $M$ and $Y_{\mathrm{st}}$ on $U$. We can take any hypersurface $V_+$ in $U$ transverse to $Y_{\mathrm{st}}$ and tangent to $V_-$, then $(\partial M \setminus S) \cup V_+$ will be a hypersurface in $M \cup U$ transverse to $\widetilde{Y}$.

There is a lot of freedom in choosing $V_+$, we will choose one which makes the Reeb dynamic easy to understand. In fact we will make the surface quadratic in shape:

$$V_+ = \left\{ \sum_1^k (b_i x_i^2 - b_i' y_i^2) + \sum_{k+1}^n a_j (x_j^2 + y_j^2) = c > 0 \right\}.$$

Such a $V_+$ is said to be a *standard contact handle.*

Let $\phi$ be the function $\sum_1^k (b_i x_i^2 - b_i' y_i^2) + \sum_{k+1}^n a_j (x_j^2 + y_j^2)$.

**Proposition 3.2.** *For positive constants $\{b_i, b_i', a_j\}$ the level set $V_+$ is everywhere transverse to the Liouville vector field $Y_{\mathrm{st}}$.*

*Proof.*

$$\begin{aligned} L_{Y_{\mathrm{st}}} \left( \sum_1^k (b_i x_i^2 - b_i' y_i^2) + \sum_{k+1}^n a_j (x_j^2 + y_j^2) \right) &= \\ \sum_1^k (4 b_i x_i^2 + 2 b_i' y_i^2) + \sum_{k+1}^n a_j (x_j^2 + y_j^2) &> 0. \end{aligned}$$

□

Strictly speaking this level set $V_+$ intersects $V_-$ at some angle. In order to have a smooth surface after handle attachment we round the corners in an arbitrarily small neighborhood of the intersection. As shown in [Yau04], the smoothing will not affect the final analysis of Reeb orbits and their Conley-Zehnder indices.

Suppose $\gamma$ is a contractible Reeb orbit, then a choice of a spanning disk $D$ and a symplectic trivialization of $\xi$ on $D$ induces a trivialization of $\xi$ along $\gamma$. The Reeb flow $\Phi_R^t$ preserves the contact structure $\xi$. Hence with respect to the chosen trivialization, the linearized Reeb flow $d(\Phi^t)$ defines a path in the symplectic matrices $Sp(2n-2)$.

The Conley-Zehnder index of a path $\Gamma$ in $Sp(2n-2)$, $\mu(\Gamma)$, is defined by the intersection number of the path with the Maslov cycle. If $\Gamma$ is a path in $Sp(2m_1)$ and $\Gamma'$ a path in $Sp(2m_2)$, then $\mu(\Gamma \oplus \Gamma') = \mu(\Gamma) + \mu(\Gamma')$.

By continuity the Conley-Zehnder index is the same for two different trivializations over the same disk. For a different spanning disk $D'$, the Conley-Zehnder index with respect to $D'$ differ from that of $D$ by twice the first Chern class of $\xi$ on the sphere bounded by $D$ and $D'$. Since we assume $c_1(\xi) = 0$, the Conley-Zehnder index of every contractible Reeb orbit is well defined and independent of choices.

Consider the Reeb dynamic on a standard handle

$$V = \left\{ \sum_1^k (b_i x_i^2 - b_i' y_i^2) + \sum_{k+1}^n a_j (x_j^2 + y_j^2) = c > 0 \right\}.$$

The Hamiltonian vector field of $\phi$, $X_\phi$, is defined by $\iota_X \omega = -d\phi$. When restricted to the level set $V$ of $\phi$, the Hamiltonian field $X$ is a multiple of the Reeb field,

$$X = \sum_1^k \left( 2 b_i x_i \partial_{y_i} + 2 b_i' y_i \partial_{x_i} \right) + \sum_{k+1}^n a_j \left( 2 x_j \partial_{y_j} - 2 y_j \partial_{x_j} \right).$$

The Reeb field on $V$ is

$$R = \frac{X}{\alpha_{\mathrm{st}}(X)} = \frac{X}{\sum_1^k (4b_i x_i^2 + 2b_i' y_i^2) + \sum_{k+1}^n a_j(x_j^2 + y_j^2)} = \frac{X}{\sum_1^k (3b_i x_i^2 + 3b_i' y_i^2) + c}.$$

Since Hamiltonian field $X$ and Reeb field $R$ have the same integral curves we look for close orbits of $X$. The Hamiltonian flow is hyperbolic in the $(x_i, y_i)$ components, and rotational with constant angular velocity in the $(x_j, y_j)$ components. Therefore any close Reeb orbit has $(x_i, y_i) = 0$ for $i \le k$ and lies on the ellipsoid $\sum_{k+1}^n a_j(x_j^2 + y_j^2) = c$. If we choose $\{a_j\}$ to be rationally independent, then the rotations in different $(x_j, y_j)$ factors will always be out of phase. Hence the only close orbits are those lying entirely on $(x_j, y_j)$-planes, i.e., multiple covers of the circle $a_j(x_j^2 + y_j^2) = c$. A simple orbit has period $\pi/a_j$ with action $c\pi/a_j$.

We can calculate the Conley-Zehnder index for each one of these Reeb orbits $\gamma$. Along $\gamma$, $R = X/c$ is a constant factor of Hamiltonian field, furthermore near $\gamma$, $R = X/c$ up to second order. Therefore with a fixed trivialization, the linearized Reeb flow represents the same path of symplectic matrices in $Sp(2n-2)$ as the linearized Hamiltonian flow, except reparametrized by a constant factor $c$.

The disk $D = \{a_j(x_j^2 + y_j^2) \le c, x_i = y_i = 0, i \ne j\}$ can be used as a spanning disk. The standard symplectic trivialization of $\mathbb{C}^n$ by coordinate vectors $\{\partial_{x_i}, \partial_{y_i}\}$ over $\gamma$ extends to the standard trivialization of $T\mathbb{C}^n$ over $D$. Identify $T\mathbb{C}^n$ as the sum of the contact distribution $\xi$ and a trivial symplectic bundle spanned by $R$ and $Y_{\mathrm{st}}$. The linearized Hamiltonian flow fixes $R|_\gamma$ and $Y_{\mathrm{st}}|_\gamma$. Suppose we pick a trivialization of $\xi$ on $D$, with respect to this direct sum trivialization of $T\mathbb{C}^n|_\gamma = \xi \oplus \mathbb{R}Y \oplus \mathbb{R}R$, the path of matrices has the form $\Gamma = \Gamma_1 \oplus \Gamma_2$, where $\Gamma_1$ is a path in $Sp(2n-2)$ which computes the Conley-Zehnder index of the Reeb orbit $\gamma$, and $\Gamma_2$ is the constant path in $Sp(2)$. Therefore $\mu(\gamma, \xi) = \mu(\gamma, T\mathbb{C}^n)$.

With respect to the standard trivialization of $T\mathbb{C}^n$, the linearized Hamiltonian flow is a block diagonal matrix

$$d(\Phi_X^t) = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{bmatrix}$$

with $2 \times 2$ blocks

$$D_i = \begin{bmatrix} \cosh 2\sqrt{b_i b_i'}t & \sqrt{\frac{b_i'}{b_i}} \sinh 2\sqrt{b_i b_i'}t \\ \sqrt{\frac{b_i}{b_i'}} \sinh 2\sqrt{b_i b_i'}t & \cosh 2\sqrt{b_i b_i'}t \end{bmatrix}, \ i \le k,$$

$$D_j = \begin{bmatrix} \cos(2a_j t) & \sin(2a_j t) \\ \sin(2a_j t) & \cos(2a_i t) \end{bmatrix}, \ j > k.$$

The Conley-Zehnder index of this path is then the sum of the Conley-Zehnder indices for the individual blocks.

For $i \le k$, the $2 \times 2$ matrices always have real eigenvalues, therefore the Conley-Zehnder index is 0.

For $i > k$, suppose $a_n$ is much larger than the others, then for any orbit in the $(x_j, y_j)$-plane, $j < n$, in the time for one revolution, the linearized flow already made a large number of rotations in the $(x_n, y_n)$ component. Therefore the Conley Zehnder index would be large. For the simple orbit in $(x_n, y_n)$-plane, the linearized flow made one full revolution in $(x_n, y_n)$ component, contributing 2 to the index, and in the other components a small positive rotation of angle less than $\pi$, contributing 1 to the index. Therefore the Conley-Zehnder index is $2 + (n - k - 1) = n - k + 1$. In fact by choosing $a_n$ sufficiently large so that $m/a_n < 1/a_j$ for all $k < j < n$, we see that an $m$-fold cover

of this orbit has index $2m+(n-k-1)$. We summarize the discussion on the Reeb dynamic as follows:

**Lemma 3.3.** *Given any positive integer $N$, in a standard $2n$ dimensional handle of Morse index $k<n$, there exist a standard contact handle $V$ such that the closed Reeb orbits of Conley-Zehnder index less than $N$ on $V$ are multiple covers of a single non-degenerate orbit $\gamma$, they have Conley–Zehnder indices $2m+(n-k-1)$ where $m$ is the multiplicity.*

This deals with orbits lying entirely in a single handle. Lemma 4.2 of [Yau04] deals with Reeb orbits of passing through multiple handles.

**Lemma 3.4.** *If $M$ is subcritical, then given any positive integer $N$, by thinning the handles, i.e., changing the quadratic shape parameter inside each standard subcritical handle, there is a hypersurface $V$ such that each contractible Reeb orbit on $V$ passing through multiple handles has Conley-Zehnder index larger than $N$.*

Let $M=M'\times\mathbb{C}$, $V'=\{\phi=c\}$ be a level set of a function $\phi$ for which $Y'$ is gradient–like, and $\phi(p)<c$ for all critical points $p$ of $Y'$.

The stabilization of a standard contact handle is a standard subcritical contact handle. By increasing the constant $\kappa$, we can simultaneously "thin" all subcritical handles in $V$, the stabilization of $V'$.

Therefore Lemmas 3.3 and 3.4 imply

**Theorem 3.5.** *Let $M=M'\times\mathbb{C}$ be a subcritical Stein manifold, Then for any $N>0$, there exists $\kappa$ sufficiently large so that each Reeb orbit of Conley–Zehnder index less than $N$ on the stabilization $V\subset M$ is an $m$–fold cover of a distinguished simple Reeb orbit $\{x^2+y^2=c-\phi(p)\}$ over an index $k$ critical point $p$ of $M'$. This orbit has Conley–Zehnder index $2m+n-k+1$.*

By the dimension formula, we see that there is no index $-1$, 0 or 1 holomorphic planes, therefore the cylindrical contact homology of $V$ is well defined.

## 4. Complex structure and $S^1$–invariant curves

Let $J'$ be a compatible complex structure on $M'$. As remarked earlier, the product complex structure $J'\oplus i$ will not be compatible with the cylindrical end structure $V\times[0,\infty)$, even though $J'\oplus i$ tames $\omega'+dx_ndy_n$. This is due to the fact that the splittings $TM|_V=TM'\oplus T\mathbb{C}$ and $TM|_V=\xi\oplus\mathbb{R}Y\oplus\mathbb{R}R$ do not match at a generic point on $V$, thus $J'\oplus i$ does not preserve the contact distribution on $V$, nor pairs the Liouville field $Y$ with the Reeb field $R$.

Let $M'$ be a Stein manifold, $f'$ a plurisubharmonic Morse function, and $Y'$ the associated complete Liouville vector field. Given any hypersurface $V'$ transverse to $Y'$ enclosing all the critical points, $f'$ can be modified to a function $\phi'$ outside a neighbourhood of the critical points such that $V'=\{\phi'=c\}$ is a level set and $Y'$ is gradient–like with respect to $\phi'$. Let $M=M'\times\mathbb{C}$ be subcritical, $\phi=\phi'+\kappa|z|^2$, and $V=\{\phi=c\}$ a stabilization of $V'$. Identify $M'=M'\times\{0\}$, $V'=V'\times\{0\}$, and let $N'=\{\phi'<c\}\subset M'\subset M$. Let $\rho$ be the projection $M\to M'$. Then $V\setminus V'\xrightarrow{\rho}N'$ is a circle bundle over $N'$.

Let $J$ be a compatible almost complex structure on $M$ with the following properties:

(1) $J$ is invariant under the $S^1$–action of rotation in the $\mathbb{C}$ factor.
(2) On $M'$, we require $J=J'\oplus i$ where $J'$ is a compatible complex structure for $M'$ and $i$ the complex multiplication in $T\mathbb{C}$. This does not conflict with compatibility since on

$V' = M \cap V'$, the splitting of $TM$ satisfies

$$\begin{aligned} \xi &= \xi' \oplus T\mathbb{C} \\ R &= R' \\ Y &= Y' \end{aligned}$$

(3) Inside each standard handle there is a single critical point $p$. For a standard handle, we will use the standard $\mathbb{R}^{2n} = \mathbb{C}^{n-1} \times \mathbb{C}$ coordinates, where $\mathbb{C}^{n-1}$ are coordinates of a handle in $M'$, and $(x_n, y_n)$ is the coordinate of the $\mathbb{C}$ factor. Let

$$\phi = \sum_1^k (b_i x_i^2 - b_i' y_i^2) + \sum_{k+1}^n a_j (x_j^2 + y_j^2).$$

Near the center of the handle the contact–type hypersurface $V$ is given by a level set $\{\phi = c\}$. The distinguished Reeb orbit $\gamma$ on $V$ is the circle $\{x_n^2 + y_n^2 = c, x_1 = y_1 = \cdots = x_{n-1} = y_{n-1} = 0\}$. On $\gamma$, the splitting of $TM = \xi \oplus (\mathbb{R}Y \oplus \mathbb{R}R)$ exactly coincides with $TM = T\mathbb{C}^{n-1} \oplus T\mathbb{C}$. Define $J|_\xi$ to be the standard complex structure on $T\mathbb{C}^{n-1}$, and $J$ pairs the Liouville vector $Y$ with the Reeb vector $R$ on $T\mathbb{C}$. The positive translation along $Y$ then extends $J$ to the region $\{x_n^2 + y_n^2 \geq c, x_1 = y_1 = \cdots = x_{n-1} = y_{n-1} = 0\}$. Then $J$ can be extended to the entire vertical plane $\{x_1 = y_1 = \cdots = x_{n-1} = y_{n-1} = 0\}$ such that $J$ preserves the splitting $TM = T\mathbb{C}^{n-1} \oplus T\mathbb{C}$ and acts by standard complex multiplication on the first factor. In this way the plane $\{x_1 = y_1 = \cdots = x_{n-1} = y_{n-1} = 0\}$ is a $J$-holomorphic.

(4) We also specify $J$ in a small neighbourhood of the plane $\{x_1 = y_1 = \cdots = x_{n-1} = y_{n-1} = 0\}$. Let $U \subset V$ be a sufficiently small neighbourhood of the Reeb orbit $\gamma$ such that $d\rho$, the projection of $TM = T\mathbb{C}^{n-1} \oplus T\mathbb{C}$ to the first factor is an isomorphism from $\xi$ to $T\mathbb{C}^{n-1}$. Define $J|_\xi$ to be the lift of the standard complex multiplication on $T\mathbb{C}^{n-1}$ to $\xi$, in other words, $J(v) = d\rho^{-1}(i \cdot d\rho(v))$. Again define $J(Y) = R$ on $U$ and extend $J$ by positive translation along the Liouville field $Y$. To extend $J$ to the interior, foliate the space between $U$ and the hypersurface $\{x_n^2 + y_n^2 = \epsilon, x_1^2 + y_1^2 + \cdots + x_{n-1}^2 + y_{n-1}^2 < \delta\}$ by quadratic hypersurfaces such that each leaf is transverse to $Y$, so each leaf is of contact type, and such that the projection $d\rho$ is an isomorphism from the induced contact distribution on each leaf to $T\mathbb{C}^{n-1}$. Define $J$ on each leaf as the lift of complex multiplication on $T\mathbb{C}^{n-1}$. Finally $J$ is the standard complex structure on $\mathbb{C}^n$ in the region $\{x_n^2 + y_n^2 < \epsilon, x_1^2 + y_1^2 + \cdots + x_{n-1}^2 + y_{n-1}^2 < \delta\}$.

Denote $\partial_\theta = x_n \partial_{y_n} - y_n \partial_{x_n} \in T\mathbb{C} \subset TM$ to be the vector field generating the $S^1$-rotation. By symmetry, $d\rho(J\partial_\theta|_V) = Z$ for some vector field $Z$ on $N'$. Let $\partial_r = x_n \partial_{x_n} + y_n \partial_{y_n}$. Suppose $u(s,t) : \mathbb{R} \times S^1 \to V \times \mathbb{R}$ is an $S^1$–invariant holomorphic cylinder in the symplectization $V \times \mathbb{R}$, asymptotic to an $m$-fold cover of a simple Reeb orbit at each end. Then up to $\mathbb{R}$–translation, $u$ is uniquely determine by the projection $u'(s,t) : S^1 \times \mathbb{R} \to V$. Furthermore $u'(s,t)$ projects to a trajectory of $Z$ on $N'$.

**Lemma 4.1.** *The vector field $Z$ is gradient–like for $\phi'$, and $\rho \circ u'$ is a trajectory of $Z$.*

*Proof.* Since $\partial_\theta$ is tangent to $V$, in the decomposition $TM|_V = \xi \oplus \mathbb{R}Y \oplus \mathbb{R}R$, we have

$$\partial_\theta = \partial_\theta - \alpha(\partial_\theta)\cdot R \oplus 0\cdot Y \oplus \alpha(\partial_\theta)\cdot R \tag{1}$$

$$= \partial_\theta - \frac{1}{2}(x_n^2+y_n^2)\cdot R \oplus 0\cdot Y \oplus \frac{1}{2}(x_n^2+y_n^2)\cdot R \tag{2}$$

$$J\partial_\theta = J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)\cdot R\right) \oplus -\frac{1}{2}(x_n^2+y_n^2)\cdot Y \oplus 0\cdot R \tag{3}$$

The projection to $V\times\{0\}$ ignores the $\mathbb{R}Y$ coordinate,

$$\begin{aligned} -\frac{\partial u'}{\partial s} &= J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R\right) \\ Z &= d\rho\left(\frac{\partial u'}{\partial s}\right) \\ &= d\rho\left(J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R\right)\right) \end{aligned}$$

Since $J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R\right)\in\xi$,

$$\begin{aligned} \alpha\left(J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R\right)\right) &= 0 \\ d\phi\left(J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R\right)\right) &= 0 \\ J\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R\right) &= Z - \frac{2\alpha'(Z)}{x_n^2+y_n^2}\partial_\theta - \frac{d\phi'(Z)}{2\kappa(x_n^2+y_n^2)}\partial_r \end{aligned}$$

The Reeb vector field $R$ is given by

$$\begin{aligned} R &= \frac{X_\phi}{\alpha(X_\phi)} \\ &= \frac{X_{\phi'} + X_{\kappa(x_n^2+y_n^2)}}{\alpha'(X_{\phi'}) + \frac{1}{2}(-y_n dx_n + x_n dy_n)(X_{\kappa(x_n^2+y_n^2)})} \\ &= \frac{X_{\phi'} + 2\kappa\partial_\theta}{\alpha'(X_{\phi'}) + \kappa(x_n^2+y_n^2)} \end{aligned}$$

Since $\omega = \omega' + dx_n dy_n$ tames $J$, if $\partial_\theta \neq \frac{1}{2}(x_n^2+y_n^2)R$, i.e., over a critical point of $\phi'$, then

$$\begin{aligned} 0 &< \omega\left(\partial_\theta - \frac{1}{2}(x_n^2+y_n^2)R,\quad Z - \frac{2\alpha'(Z)}{x_n^2+y_n^2}\partial_\theta - \frac{d\phi'(Z)}{2\kappa(x_n^2+y_n^2)}\partial_r\right) \\ &= -\frac{x_n^2+y_n^2}{2\alpha(X_\phi)}\omega'(X_{\phi'}, Z) + \frac{\alpha'(X_{\phi'})d\phi'(Z)}{2\alpha(X_\phi)\kappa(x_n^2+y_n^2)}dxdy(\partial_\theta, \partial_r) \\ &= d\phi'(Z)\left(\frac{x_n^2+y_n^2}{2\alpha(X_\phi)} + \frac{\alpha'(X_{\phi'})}{2\alpha(X_\phi)\kappa}\right) \end{aligned}$$

Thus $d\phi'(Z) > 0$ for $Z \neq 0$. The zeros of $Z$ coincides with the critical points of $\phi'$. Near a critical point, since the complex structure is defined to be the lift of the standard complex structure on the base, it is not hard to find $Z$ explicitly and see that it is indeed gradient–like for $\phi'$.

For an $S^1$-invariant holomorphic cylinder $u$, by $S^1$-invariance, $du(\partial_t) = m\partial_\theta$ if $u$ is asymptotic to an $m$-fold cover of a simple Reeb orbit. So $du(-\partial_s) = mJ\partial_\theta$, and $\rho\circ u'$ is a trajectory of $Z$. □

This covers invariant cylinders in the symplectization, the invariant planes in the cobordism $M$ are also govern by the vector field $Z$.

**Lemma 4.2.** *The vector field $J\partial_\theta$ is Morse–Bott with zero set $M'$.*

*Proof.* By the construction of $J$, $J\partial_\theta = -rdr$ on $M'$. Also $\omega(\partial_\theta, J\partial_\theta) > 0$, $(\iota_{\partial_\theta}\omega)(J\partial_\theta) > 0$. But $\iota_{\partial_\theta}\omega = -(xdx + ydy) = d(-\frac{1}{2}(x^2 + y^2))$. Hence $J\partial_\theta$ is gradient–like with respect to the function $-\frac{1}{2}(x^2 + y^2)$. □

**Lemma 4.3.** *Each trajectory of the positive flow of $J\partial_\theta$ is asymptotic to a point on $M'$.*

*Proof.* We only need to check that a trajectory does not escape to infinity without hitting $M'$. On the cylindrical end $V \times [0, \infty)$, Equation 3 shows that $J\partial_\theta$ is always pointing towards the compact filling. It follows that any positive trajectory of $J\partial_\theta$ stays within a compact set of $M$. Since $J\partial_\theta$ is gradient–like, each trajectory must approach a zero of the vector field. □

**Lemma 4.4.** *Each trajectory of the negative flow of $J\partial_\theta$ eventually enters the cylindrical end $V \times [0, \infty)$.*

*Proof.* In fact a negative trajectory must leave any compact region. Since the negative flow stays away from the zero set, if a trajectory stays within a compact region then there is an accumulation point where $J\partial_\theta \neq 0$, a contradiction since $J\partial_\theta$ is gradient–like. □

**Theorem 4.5.** *$M$ is foliated by simple $S^1$-invariant holomorphic planes.*

*Proof.* Given a point $p = (x, re^{i\theta}) \in M' \times \mathbb{C}$, let $\Phi(\tau)$ denote the flow of $J\partial_\theta$ by time $\tau$ starting at $p$. Define

$$u(s, t) = e^{it} \circ \Phi(-s),$$

where $e^{it} \circ (x, re^{i\theta}) = (x, re^{i(\theta+t)})$ is the $S^1$ action. By Lemma 4.3, as $s \to -\infty$, the cylinder is asymptotic to a point on $M'$. Therefore by removal of singularity, $u$ can be extended to a map of a holomorphic plane to $M$. As $s \to \infty$, by Lemma 4.4, the cylinder enters the cylindrical end $V \times [0, \infty)$, where it is governed by the flow of $Z$. In particular, since $-J\partial_\theta$ has a positive $Y$ component, $u \cap V$ is either empty, a point on $V'$, or a single circle on $V \setminus V'$. The part of $u$ in the cylindrical end $V \times [0, \infty)$ projects to a semi–infinite trajectory of $Z$. If this trajectory lies on the unstable manifold of some critical point $p_i$, then $u$ is asymptotic to the Reeb orbit $\gamma_i$ over the critical point $p_i$. Hence there is an $S^1$-invariant holomorphic plane through every point of $M \setminus M'$ and therefore all of $M$. □

If two $S^1$-invariant holomorphic planes intersect, then by $S^1$ symmetry they intersect in at least a circle and therefore coincide. Hence the planes in Theorem 4.5 and their multiple covers are in fact all the $S^1$-invariant holomorphic planes.

Consider a point $q$ on $N' \subset M' \times 0 \subset M$. There is a unique simple $S^1$-invariant holomorphic plane $u_q$ through $q$, intersecting $V$ in a single circle. By the projection $\rho$, this circle maps onto a point $q' \in N'$. In this way have a diffeomorphism $\pi : N' \to N'$. Since the vertical plane over a critical point $p$ is holomorphic by construction, $\pi(p) = p$. Let $U_p$ and $S_p$ denote the unstable and stable manifolds of a critical point $p$ with respect to the vector field $-Z$. If $\pi(q)$ is on the stable manifold of a critical point $p$, then $u_q$ is asymptotic to $\gamma_p$, the Reeb orbit over $p$. If $p$ is a critical point of $\phi'$ of Morse index $k$, then the family of $S^1$-invariant planes asymptotic to $\gamma_p$ is $2n - k - 2$ dimensional, and is parametrized by $S_p$. There is exactly one holomorphic plane asymptotic to $\gamma_p$ passing through $\pi^{-1}(U_p)$, namely the vertical plane over $p$.

## 5. Transversality

The proof of the following theorem does not yet exist in print. However the theorem follows from the polyfold theory of Hofer–Wysocki–Zehnder [HWZ06] [HWZ07] together with the work of McDuff [McD07] [MT06], which proved the corresponding result for moduli spaces of closed curves in the setting of Gromov–Witten theory. In particular, in section 4.2.2 of [MT06], we can replace the regularization process of Liu–Tian [LT98] by the polyfold regularization process of Hofer–Wysocki–Zehnder.

The idea is that if a component of a moduli space does not have a $S^1$-invariant element, then regularity can be achieved by performing an abstract perturbation on a slice transverse to the $S^1$ orbits, and then extending this perturbation by the $S^1$-action.

**Theorem 5.1.** *Let $M$ be a symplectic cobordism together with a compatible complex structure $J$ which is invariant under an $S^1$-action. Let $K > 0$ be a real number such that all Reeb orbits with action $\alpha(\gamma)$ less than $K$ are non-degenerate and invariant under the $S^1$-action. If $\sum \gamma_i^+ < K$ and a component of the moduli space $\mathcal{M}_{g,\{\gamma_1^+ \dots, \gamma_a^+\},\{\gamma_1^- \dots, \gamma_b^-\},m}$ does not contain an $S^1$-invariant element, then after perturbation $\mathcal{M}^{virt}$ can be realized as a weighted branched manifold with boundary with a free $S^1$-action such that the evaluation map $ev : \mathcal{M}^{virt} \to \overline{M}^m$ is equivariant.*

Note that by taking $\kappa$ sufficiently large, all orbits of small action are multiple covers of the distinguished orbits.

In the symplectization $V \times \mathbb{R}$, the following is proved in [Yau04].

**Theorem 5.2.** *The simple holomorphic cylinder $u$ corresponding to a trajectory of $Z$ between two critical point of neighboring indices and its multiple covers are regular.*

It follows that the differential for $HC^{\mathsf{cyl}}$ is precisely the Morse differential for the vector field $Z$, and $HC^{\mathsf{cyl}}$ is a direct sum of copies of $H^*(M') = H^*(M)$, where the Reeb orbits of multiplicity $i$ contributes a copy of $H^*(M)$ for each $i \geq 1$. Note that *cohomology* of $M$ is computed because $Z$ needs to be reversed to be a negative gradient.

**Theorem 5.3.** *The full contact homology of $V$ is isomorphic to $\Lambda\left(HC^{\mathsf{cyl}}\right)$.*

*Proof.* If $\Sigma$ has multiple negative punctures, then any $u : \Sigma \to M \times \mathbb{R}$ in the compactified moduli space, which includes nodal and multistory curves, does not admit an $S^1$-family of automorphisms. Hence by Theorem 5.1, the regularized moduli space, if non-empty, has a free $S^1$-action as well as the $\mathbb{R}$-translation. However, the expected dimension of the moduli space is 1, thus $\mathcal{M}$ is empty. Therefore the differential in the full contact homology complex arises from the count of holomorphic cylinders only. Therefore $HC(V)$ is just the graded commutative polynomial algebra with $HC^{\mathsf{cyl}}$ as the linear part. □

**Theorem 5.4.** *The simple holomorphic plane over a critical point is regular.*

*Proof.* Since the complex structure near the plane is known, we directly compute $D_u$, the linearized $\overline{\partial}$ operator at $u$. In local coordinates,

$$D_u v = ds - J(u)\eta dt$$

where $v = (\mu_1, \nu_1, \dots, \mu_n, \nu_n) \in u^*(TM)$ and

$$\eta = \frac{1}{2}(\partial_s v + J(u)\partial_t v + (\partial_v J)(u)\partial_t(u))$$

Transversality is equivalent to the surjectivity of the differential operator $D_u : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

At $(t,0,\dots,0,x_n(t),y_n(t))\in V$, we have

$$Y(t)=\begin{cases} 2t\partial_{x_1}+\frac{1}{2}x_n(t)\partial_{x_n}+\frac{1}{2}y_n(t)\partial_{y_n}, & \text{if Morse index is zero}\\ \frac{1}{2}t\partial_{x_1}+\frac{1}{2}x_n(t)\partial_{x_n}+\frac{1}{2}y_n(t)\partial_{y_n}, & \text{if Morse index is greater than zero}\end{cases}$$

$$R(t)=\begin{cases} \frac{1}{3b_1t^2+c}(2b_1t\partial_{y_1}-2\kappa y_n(t)\partial_{x_n}+2\kappa x_n(t)\partial_{y_n}), & \text{if Morse index is zero}\\ \frac{1}{c}(2a_1t\partial_{y_1}-2\kappa y_n(t)\partial_{x_n}+2\kappa x_n(t)\partial_{y_n}), & \text{if Morse index is greater than zero}\end{cases}$$

In any case,

$$\begin{aligned} Y(t) &= O(t)\partial_{x_1}+(\text{constant}+O(t^2))\partial_{x_n}+(\text{constant}+O(t^2))\partial_{y_n}\\ R(t) &= O(t)\partial_{y_1}+(\text{constant}+O(t^2))\partial_{x_n}+(\text{constant}+O(t^2))\partial_{y_n}\end{aligned}$$

For any tangent vector $v$, $v$ decomposes into $\xi\oplus\mathbb{R}Y\oplus\mathbb{R}R$ as follows,

$$v=v-\frac{\partial_v\phi}{\partial_Y\phi}Y-\alpha(v)R\oplus\frac{\partial_v\phi}{\partial_Y\phi}Y\oplus\alpha(v)R$$

The contact distribution contains $\{\partial_{x_2},\partial_{y_2},\dots,\partial_{x_{n-1}},\partial_{y_{n-1}}\}$. By definition, $J$ is the standard complex structure on $\{\partial_{x_2},\partial_{y_2},\dots,\partial_{x_{n-1}},\partial_{y_{n-1}}\}$.

Furthermore

$$\begin{aligned} \partial_{x_1} &= \partial_{x_1}-O(t)Y-O(t)R\oplus O(t)Y\oplus O(t)R\\ \partial_{y_1} &= \partial_{y_1}-O(t)Y-O(t)R\oplus O(t)Y\oplus O(t)R\\ \partial_{x_n} &= \partial_{x_n}-(\text{constant}+O(t^2))Y-(\text{constant}+O(t^2))R\oplus(\text{constant}+O(t^2))Y\\ &\quad\oplus(\text{constant}+O(t^2))R\\ \partial_{y_n} &= \partial_{y_n}-(\text{constant}+O(t^2))Y-(\text{constant}+O(t^2))R\oplus(\text{constant}+O(t^2))Y\\ &\quad\oplus(\text{constant}+O(t^2))R\end{aligned}$$

Hence

$$\partial_{(1,0,\dots,0)}J=\begin{bmatrix} * & * & 0 & \dots & 0 & * & *\\ * & * & 0 & \dots & 0 & * & *\\ 0 & 0 & 0 & \dots & 0 & 0 & 0\\ \vdots & & & \ddots & & & \vdots\\ 0 & 0 & 0 & \dots & 0 & 0 & 0\\ * & * & 0 & \dots & 0 & 0 & 0\\ * & * & 0 & \dots & 0 & 0 & 0\end{bmatrix}$$

Also $\partial_t(u)=(0,\dots,0,\partial_t x_n,\partial_t y_n)$, hence

$$(\partial_{(1,0,\dots,0)}J)\partial_t(u)=\begin{bmatrix} A_1(s,t)\\ A_2(s,t)\\ 0\\ \vdots\\ 0\end{bmatrix}$$

Differentiating in the other directions yields

$$D_u \begin{bmatrix} \mu_1 \\ \nu_1 \\ \vdots \\ \mu_n \\ \nu_n \end{bmatrix} = \begin{bmatrix} \partial_s \mu_1 - \partial_t \nu_1 + A_1(s,t)\mu_1 + B_1(s,t)\nu_1 \\ \partial_s \nu_1 + \partial_t \mu_1 + A_2(s,t)\mu_1 + B_2(s,t)\nu_1 \\ \vdots \\ \partial_s \mu_n - \partial_t \nu_n + A_{2n-1}(s,t)\mu_n + B_{2n-1}(s,t)\nu_n \\ \partial_s \nu_n + \partial_t \nu_n + A_{2n}(s,t)\mu_n + B_{2n}(s,t)\nu_n \end{bmatrix}$$

The operator $D_u$ splits into a direct sum of Cauchy–Riemann operators on $\mathbb{C}$. Therefore $D_u$ is surjective. $\square$

By an identical argument, we also have

**Theorem 5.5.** *A multiple covers of the simple holomorphic plane over a critical point is regular.*

Suppose $\sigma = [\sum_{i=1}^k \gamma_i]$ is an element of $HC^{\mathsf{cyl}}$ where each $\gamma_i$ is a simple Reeb orbit over a critical point $p_i$ of the same index $k$. By Poincaré duality, $\sigma \in H^*(M') \cong H_*(M', \partial M')$ is represented by the union of stable manifolds $\bigcup_{i=1}^k S_p$. If $\theta$ is a compactly supported form on $M = M' \times \mathbb{C}$, then integration along fibers is an isomorphism, and we obtain $\theta'$, a compactly supported form on $M'$.

**Lemma 5.6.** *If $\sigma = [\sum_{i=1}^k \gamma_i] \in HC^{\mathsf{cyl}}$ consists of simple orbits and $\theta \in H^*(M, \partial M)$, then*

$$\int_{\mathcal{M}_{[\sigma]}} ev^*(\theta) = \int_{\bigcup_{i=1}^k S_p} \theta'$$

*Proof.* The Poincaré dual of $\theta$ can be represented by $\pi(\beta)$, where $\beta$ is a union of unstable manifolds of the negative gradient–like vector field $-Z$, and $\pi$ is the diffeomorphism defined in the previous section. The correlator $\int_{\mathcal{M}_{[\sigma]}} ev^*(\theta)$ counts the discrete number of holomorphic planes asymptotic to some orbit $\gamma_i$ passing through $\pi(\beta)$. Suppose $u = (u', f) : \mathbb{C} \to M = M' \times \mathbb{C}$ is a holomorphic plane with one marked point, which we assume to be 0 without loss of generality. Let $\mathcal{M}_{\{[\sigma], \pi(\beta)\}}$ be the moduli space $\{u \in \mathcal{M}_{[\sigma]} : u(0) \in \pi(\beta)\}$. Since $\pi(\beta) \subset M'$ lies on the fixed point set, $S^1$ acts on $\mathcal{M}_{\{[\sigma], \pi(\beta)\}}$. By Theorem 5.1, after regularization, components of $\mathcal{M}_{\{[\sigma], \pi(\beta)\}}$ without any $S^1$-invariant elements vanish for dimensional reason. By construction, the $S^1$-invariant elements of $\mathcal{M}_{[\sigma], \beta}$ consist of vertical planes of critical points. These planes are already regular by Theorem 5.4, each contributing 1 or $-1$ depending on orientation. Therefore $\int_{\mathcal{M}_{[\sigma]}} ev^*(\theta)$ is precisely the intersection number between the Poincaré dual of $\theta$ (regarded as a submanifold of M') and $\bigcup_{i=1}^k S_p$ in $M'$. $\square$

**Lemma 5.7.** *If $\sigma = [\sum_{i=1}^k \gamma_i] \in HC^{\mathsf{cyl}}$ consists of orbits of multiplicity $m$ and $\theta \in H^*(M, \partial M)$, then*

$$(m-1)! \int_{\mathcal{M}_{[\sigma]}} ev^*(\theta) \psi^{m-1} = \int_{\bigcup_{i=1}^k S_p} \theta'$$

*Proof.* Let $u = (u', f)$ as before. Since the complex structure $J$ is split on $M'$, $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at 0. We claim that the descendant

$$(m-1)! \int_{\mathcal{M}_{[\sigma]}} ev^*(\theta) \psi^{m-1}$$

is the number of holomorphic planes in $\mathcal{M}_{\{[\sigma], \pi(\beta)\}}$ such that $f$ has ramification order at least $(m-1)$ at 0.

Note that $f$ is nowhere constantly zero, because with only one marked point, the domain cannot have any ghost bubbling, and the holomorphic plane does not lie entirely on $M'$. Now

$\int_{\mathcal{M}_{[\sigma]}} ev^*(\theta)\psi^{m-1} = \int_{\mathcal{M}_{\{[\sigma],\pi(\beta)\}}} \psi^{m-1}$. Note that $df$ is a section of the tautological bundle $L$ over $\mathcal{M}_{\{[\sigma],\pi(\beta)\}}$, as it takes a tangent vector $\partial_z$ at the marked point to the number $\frac{df}{dz}(0)$. Hence $\int_{\mathcal{M}_{\{[\sigma],\pi(\beta)\}}} \psi^1$ is precisely the number of holomorphic planes where $df = 0$. For higher powers of $\psi$, consider the map

$$u \to (f'(0), f''(0), \dots, f^{m-1}(0)).$$

This is in fact a section of the bundle $L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes m-1}$ and can be seen as follows.

Under change of coordinates we have

$$\frac{d^i f}{dz^i} = \left(\frac{dz}{dw}\right)^i \frac{d^i f}{dw^i} + \text{ terms involving lower order derivatives of } f.$$

Therefor the transition matrix is lower triangular

$$\begin{bmatrix} \frac{df}{dz} \\ \frac{d^2 f}{dz^2} \\ \vdots \\ \frac{d^{m-1} f}{dz^{m-1}} \end{bmatrix} = \begin{bmatrix} \frac{dz}{dw} & 0 & 0 & \cdots & 0 \\ * & \left(\frac{dz}{dw}\right)^2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ * & * & * & \cdots & \left(\frac{dz}{dw}\right)^{m-1} \end{bmatrix} \begin{bmatrix} \frac{df}{dw} \\ \frac{d^2 f}{dw^2} \\ \vdots \\ \frac{d^{m-1} f}{dw^{m-1}} \end{bmatrix}$$

Hence the bundle is isomorphic to $L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes m-1}$. The number of holomorphic planes of ramification order at least $m-1$ is the intersection of this section with the zero section. Therefore it equals the integral of the Euler class of $L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes m-1}$ over $\mathcal{M}$. But

$$\begin{aligned} e(L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes m-1}) &= c_1(L)c_1(L^{\otimes 2}) \cdots c_1(L^{\otimes m-1}) \\ &= (\psi)(2\psi) \cdots ((m-1)\psi) \\ &= (m-1)!\psi^{m-1} \end{aligned}$$

As before, components of $\mathcal{M}_{\{[\sigma],\pi(\beta)\}}$ without any $S^1$-invariant elements can be ignored, and there exactly one $m$-fold cover of each simple plane over a critical point such that $f(0) = 0$. □

Combining Lemmas 5.6 and 5.7 yields the desired non-degenerate pairing for Theorem 1.2.

Correlators with multiple marked points can also be considered. Remember $\theta'$ denotes the form on $M'$ after integrating $\theta$ along the $\mathbb{C}$-fibers.

**Theorem 5.8.** *If $m \geq 3$ then*

$$\int_{\mathcal{M}_{[\sigma]}} ev_1^*(\theta_1) ev_2^*(\theta_2) \cdots ev_m^*(\theta_m) = 0$$

*If $m = 2$ then the only non-vanishing correlator is*

$$\int_{\mathcal{M}_{[\sigma_0]}} ev_1^*(\theta_1) ev_2^*(\theta_2) = \int_{M'} \theta_1' \theta_2'$$

*where $\sigma_0$ is the generator of $HC^{\mathsf{cyl}}$ corresponding to a simple Reeb orbit over the index 0 critical point.*

*Proof.* Let $\beta_i$ be cycles on $M'$ representing the Poincaré dual of $\theta_i$. An $S^1$-invariant plane must have all its marked points on $M'$. The only possible configuration for the domain is a copy of $\mathbb{C}$ with a tree of ghost bubbles attached at the origin, and all marked point lie on the ghost bubble tree. The map on $\mathbb{C}$ is a $S^1$-invariant plane with the origin mapping to $M'$. Therefore the cycles $\beta_i$ must have common intersections. However, since $M$ is subcritical, each $\beta_i$ has dimension at most

$n-1$. Since $M'$ has dimension $2n-2$, the $\beta_i$'s can perturbed within $M'$ to be disjoint unless $m=2$ and both $\beta_1$ and $\beta_2$ has maximum dimension $n-1$. Thus

$$\int_{\mathcal{M}_{[\sigma]}} ev_1^*(\theta_1)ev_2^*(\theta_2)\cdots ev_m^*(\theta_m) = 0$$

When $m=2$, $\beta_1 \cap \beta_2$ consists of points, we can perturb $\beta_1$ on $M'$ so that $\beta_1 \cap \beta_2$ lies in a neighborhood of the index 0 critical point on $M'$ and each is a transverse intersection. If $\sigma$ consists of multiplicity $m$ Reeb orbits over critical points of index $k$, then the $S^1$-invariant elements asymptotic to $\sigma$ are mapped to the unstable manifold of the critical points by the evaluation map. Hence unless $k=0$, there will be no $S^1$-invariant holomorphic planes passing through $\beta_1 \cap \beta_2$ and asymptotic to $\sigma$. If $\sigma = \sigma_0$ then because the holomorphic planes near the vertical plane over the index 0 critical point are also regular, the number of $S^1$-invariant planes is the algebraic number of $\beta_1 \cap \beta_2$,

$$\int_{\mathcal{M}_{[\sigma_0]}} ev_1^*(\theta_1)ev_2^*(\theta_2) = \int_{M'} \theta_1'\theta_2'$$

□

## 6. Subcritical polarizations

A *polarized Kähler manifold* $(M, \omega, J, \Sigma)$ is a Kähler manifold $(M, \omega, J)$ with an integral Kähler form $\omega$ and a smooth reduced complex hypersurface $\Sigma$ representing the Poincaré dual of $k[\omega]$. The number $k$ is called the *degree* of the polarization.

There is a canonical plurisubharmonic function $f_\Sigma : M \setminus \Sigma \to \mathbb{R}$ associated to a polarization. Let $\mathcal{L}$ be the holomorphic line bundle associated to the divisor $\Sigma$ and $s$ the holomorphic section (unique up to scalar multiplication) of $\mathbb{L}$ whose zero section is $\Sigma$. Choose a hermitian metric $\|\cdot\|$ such that the compatible metric connection $\nabla$ have curvature $R = 2\pi i k\omega$. Then define

$$f_\Sigma = -\frac{1}{4\pi k}\log\|s(x)\|^2.$$

It is not hard to check that $f_\Sigma$ is plurisubharmonic and in fact $-dd^J f_\Sigma = \omega$. All the critical points of $f_\Sigma$ lie within a compact subset of $M \setminus \Sigma$. Note that this canonical $f$ does *not* give a complete Liouville vector field. However as we stated before we can always rescale to $e^f$ instead.

A polarization is *subcritical* if there is a plurisubharmonic Morse function $f$ such that $(M\setminus\Sigma, J, f)$ is a subcritical Stein manifold, and $f = f_\Sigma$ outside a compact subset of $M \setminus \Sigma$.

Split the $M$ along a level set $V = \|s(x)\| = \epsilon$. Let $\xi$ be the maximum complex subbundle of $TV$, we will calculate the rational contact homology of $(V, \xi)$ in two ways.

On one hand $V$ is a contact–type hypersurface of the subcritical Stein manifold $(M \setminus \Sigma, J, e^f)$. Therefore by previous computations $HC(V, \xi)$ is the free graded algebra generated by $\bigoplus_{i=1}^{\infty} H^{2n-*}(M)[2i-4]$. In particular the lowest graded piece lies in degree $2n-2-D$ and is isomorphic to $H^D(M)$, where $D < n$ is the highest degree of non-vanishing cohomology.

The other part of the splitting is biholomorphic to the complex normal bundle of $\Sigma$. Clearly $V$ is everywhere transverse to the radial vector field, since in a local coordinate chart $s(x) = cz + O(z^2)$ where $z$ is the fiber coordinate. Therefore $(V, \xi)$ is contactomorphic to the prequantization of $\Sigma$. The rational symplectic field theory of a prequantization can be computed using Morse-Bott setup of [Bou04]. We summarize the necessary results.

- The Reeb orbits of the prequantization coincides with the $S^1$ fibers of the fibration. The space of orbits consist of copies of $\Sigma$, one for each multiplicity.

- The Morse-Bott complex for rational contact homology is generated as a graded algebra by the critical points of a Morse function on each orbit space. In this case the differential is given by the Morse differential on each copy of $\Sigma$.
- To work out the grading, without loss of generality normalize the symplectic form so that $(\omega, \beta) = 1$ for some $\beta \in H_2(M)$. Take a multiplicity $k$ orbit, $\gamma_k$, above a point $p \in \Sigma$. Using the product framing the Maslov index is zero. By Lefschetz theorem $H_2(\Sigma) = H_2(M)$, since $\Sigma$ represents $k[\omega]$, there is a surface $C$ on $\Sigma$ passing through $p$ homologous to $\beta$. This surface then lifts to a section of the normal bundle with a zero of order $k$ at $p$ and no poles. This section realizes a capping surface for $\gamma_k$. The Maslov index in this trivialization will be $2(c_1(T\Sigma), \beta) := 2c$. Therefore the Maslov index for a degenerate multiplicity $l$ loops is $\frac{2cl}{k}$. The Conley-Zehnder index for a Morse index $i$ critical point with multiplicity $l$ is
$$\frac{2cl}{k} - \frac{1}{2}\dim(\Sigma) + i = i + 1 - n + \frac{2cl}{k}.$$
The contact homology grading is
$$(n-3) + i + 1 - n + \frac{2cl}{k} = i - 2 + \frac{2cl}{k}.$$

**Theorem 6.1.** *If $c_1(M)$ is proportional to $\omega$, then $M$ is uniruled.*

*Proof.* Note that $c_1(M)$ is proportional to $\omega$ means $c_1$ is supported on $\Sigma$, and $c_1(M \setminus \Sigma) = 0$, hence all our previous computations apply.

Consider the lowest graded piece of $HC(V)$. Note that this coincides with the lowest graded piece of $HC^{\mathsf{cyl}}(V)$.

Grading consideration immediately yields

$$\begin{aligned} 2n - 2 - D &= -2 + \frac{2c}{k} && (4)\\ H^D(M \setminus \Sigma) &\cong HC^{2n-2-D}(V) \cong H_0(\Sigma) \cong \mathbb{R} && (5) \end{aligned}$$

By Theorem 1.2 there is a non-trivial pairing
$$GW : H^D(M \setminus \Sigma) \otimes H^{2n-D}(M \setminus \Sigma, V) \longrightarrow \mathbb{R}.$$

This implies that the simple orbit on the prequantization actually bounds a holomorphic plane in $M \setminus \Sigma$. On the normal bundle side it bounds a fiber, these two planes topologically glued to a sphere intersecting $\Sigma$ once. Since $\Sigma$ represents $k[\omega]$, it follows that $\omega$ is primitive and $k = 1$. In fact the holomorphic plane on the normal bundle side is also generic. By the SFT gluing theorem, we have the non-vanishing of a 2-point Gromov-Witten invariant of the original Kähler manifold,
$$\sum_{\omega(A)=1} \int_{\mathcal{M}_{g=0,m=2,A}} ev_1^*(\theta_1) ev_2^*(\theta_2) = 1,$$
where $\theta_1$ is Poincaré dual to the point class and $\theta_2$ to the generator of $H_D(M \setminus \Sigma)$ (viewed as a class in $H_D(M)$ via inclusion). This means that through every generic point there is a holomorphic sphere, and $M$ is uniruled. □

**Theorem 6.2.** *Suppose $(M^{2n}, J)$ admits a subcritical polarization such that $c_1(M)$ is proportional to $\omega$. Let $a_0, \dots, a_D$ be the Betti numbers of $M \setminus \Sigma$,*
*$b_0, \dots, b_{n-2}, b_{n-1}, b_n, \dots, b_{2n-2}$ the Betti numbers of $\Sigma$. Then*

**(a):** *the sequence $\{a_i\}$ is symmetric, $a_i = a_{D-i}$,*

**(b):** *the sequences* $\{b_{2i}\}_{2i<n}$ *and* $\{b_{2i+1}\}_{2i+1<n}$ *are non-decreasing, if we define* $a_i = 0$ *for* $D < i < n$*, then*

$$b_{2i} = \sum_0^i a_{2i}, \quad b_{2i+1} = \sum_0^i a_{2i+1}.$$

*Proof.* Consider the linear part of $HC(V)$, i.e., the cylindrical contact homology. From Equation 4 and $k = 1$ we have $D = 2(n-c)$ is even. Since

$$HC^{\mathsf{cyl}}(V) \cong \bigoplus_{i=0}^{\infty} H^{2n-*}(M)[2i-4],$$

the ranks of $HC^{\mathsf{cyl},i}(V)$, starting from lowest degree $2n-2-D$, is

$$a_D,\ a_{D-1},\ a_{D-2}+a_D,\ a_{D-3}+a_{D-1},\ a_{D-4}+a_{D-2}+a_D, \ldots.$$

We can also calculate $HC^{\mathsf{cyl}}(V)$ by Morse-Bott methods, and it is the direct sum of infinitely many copies of $H_*(\Sigma)$ with a grading shift of $2ci = (2n-D)i$ for the $i$-th copy. We have Poincaré duality on $\Sigma$ so $b_i = b_{2n-2-i}$.

We equate the ranks and have equations $a_D = b_0, \cdots$. Since $2n - D > n > D$, we have

$$\begin{aligned} a_D + \ldots + a_2 &= b_{D-2} \\ a_D + \ldots + a_0 &= b_{2n-D} + b_0 = b_{D-2} + b_0 \end{aligned}$$

Hence $a_0 = a_D$.

$$\begin{aligned} a_D + \ldots + a_4 &= b_{D-4} \\ a_D + \ldots + a_0 &= b_{2n-D+2} + b_2 = b_{D-4} + b_2 \end{aligned}$$

Hence $a_0 + a_2 = b_2 = a_D + a_{D-2}$, so $a_2 = a_{D-2}$. Inductively we have $a_i = a_{D-i}$, proving **(a)**. Part **(b)** is just the first $(n-1)$ equations rewritten using **(a)**. It is not hard to check that **(a)** and **(b)** are the only relations. ∎

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